Demystifying Proof Assistants

An Introduction to Interactive Theorem Proving

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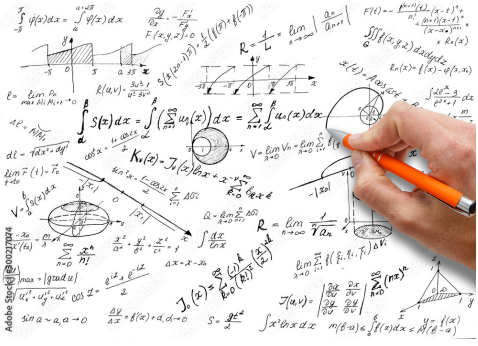
Outline

Motivation




FROM THE MAKERS OF WOLFRAM LANGUAGE AND MATHEMATICA
WolframAlpha
Enter what you want to calculate or know about
NATURAL LANGUAGE
MATH INPUT
EXTENDED KEYBOARD
EXAMPLES
UPLOAD
RANDOM

Level 1

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Thm: In a monoid M, left inverses are equal to right inverses.
Proof: Let $x, l, r \in M$ with $l \cdot x = 1 = x \cdot r$. Then

$$l = l \cdot 1 = l \cdot (x \cdot r) = (l \cdot x) \cdot r = 1 \cdot r = r$$

This concludes the proof. $\square$

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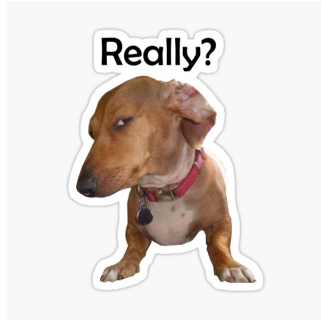

Level ∞

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INTER-UNIVERSAL TEICHMÜLLER THEORY I: CONSTRUCTION OF HODGE THEATERS

Shinichi Mochizuki

May 2020

Abstract. The present paper is the first in a series of four papers, the goal of which is to establish an *arithmetic* version of *Teichmüller theory* for **number fields** equipped with an **elliptic curve** — which we refer to as **"inter-universal Teichmüller theory"** — by applying the theory of *semi-graphs of anabelioids*, *Frobenioids*, the *étale theta function*, and *log-shells* developed in earlier papers by the author. We begin by fixing what we call *"initial* Θ*-data"*, which consists of an *elliptic curve* E_F over a *number field* F, and a *prime number* $l \geq 5$, as well as some other technical data satisfying certain technical properties. This data determines various *hyperbolic orbicurves* that are related via finite étale coverings to the once-punctured elliptic curve X_F determined by E_F. These finite étale coverings admit various *symmetry properties* arising from the **additive** and **multiplicative** structures on the ring $\mathbb{F}_l = \mathbb{Z}/l\mathbb{Z}$ acting on the *l-torsion points* of the elliptic curve. We then construct *"*$\Theta^{\pm\mathrm{ell}}$*NF-Hodge theaters"* associated to the given Θ-data. These $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters may be thought of as *miniature models of* **conventional scheme theory** in which the **two underlying combinatorial dimensions** of a number field — which may be thought of as corresponding to the **additive** and **multiplicative** structures of a ring or, alternatively, to the **group of units** and **value group** of a local field associated to the number field — are, in some sense, **"dismantled"** or **"disentangled"** from one another. All $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters are isomorphic to one another, but may also be related to one another by means of a "Θ-link", which relates certain *Frobenioid-theoretic* portions of one $\Theta^{\pm\mathrm{ell}}$NF-Hodge

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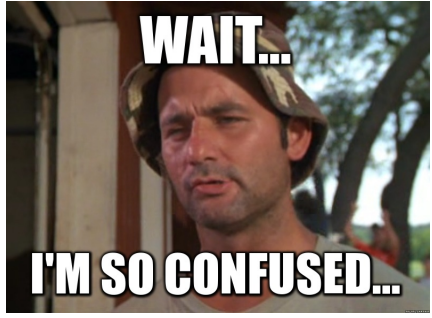

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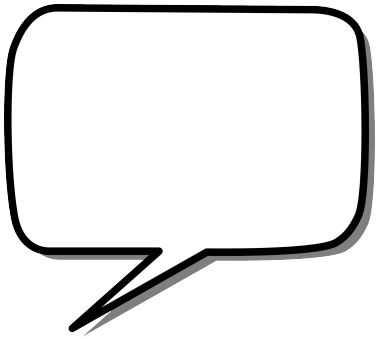

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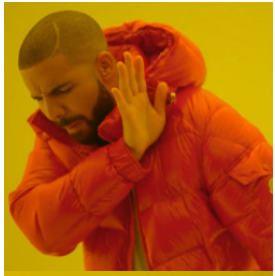
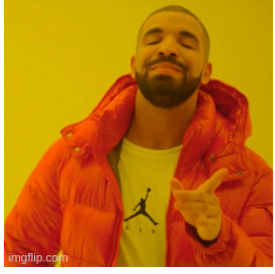


Some random mathematician on the other side of the world



Automatic Theorem Prover

Benefits

Process

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- REPL or LSP

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- highlighting, type checking

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- tactics vs term-style

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Process

- REPL or LSP
- highlighting, type checking
- tactics vs term-style
- (optional) automatic proof insertion
- (optional) program compilation
- Under the hood: (higher order) unification, term rewriting

Crash Course Type Theory

Keywords:

- Propositions as types

(What are you looking at? Please focus on the talk :P)

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Keywords:

- Propositions as types
- Higher order types
- λ calculus
- Curry-Howard isomorphism
- Proof (ir-)relevance, ex-/intensionality
- Currying (higher order functions)

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Popular Tools (Overview)

- Prolog
- Coq
- Agda
- Isabelle
- Lean

Prolog (1972)

- one of the oldest logic programming languages
- uses Horn clauses
- widely used in computer science, business and AI

```
mortal(X) :- human(X).

human(socrates).

mortal(X) :- human(X).

?- mortal(socrates).
% Yes

?- mortal(X).
% X = socrates
```

Coq (Calculus of Constructions) (1989)

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```
plus_comm =
fun n m : nat =>
nat_ind (fun n0 : nat => n0 + m = m + n0)
  (plus_n_0 m)
  (fun (y : nat) (H : y + m = m + y) =>
   eq_ind (S (m + y))
     (fun n0 : nat => S (y + m) = n0)
     (f_equal S H)
     (m + S y)
     (plus_n_Sm m y)) n
     : forall n m : nat, n + m = m + n
```

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- extensions for *cubical type theory* (HoTT)
- no built-in tactics language (dependent types)
- extensive use of Unicode characters

Agda vs Haskell

Agda:

```
_+_ : ℕ→ℕ→ℕ
m +  zero    = m
m + (succ n) = succ (m + n)
```

Haskell:

```
(+) :: Nat -> Nat -> Nat
m +  zero    = m
m + (succ n) = succ (m + n)
```

Isabelle (1986)

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- small kernel ($\rightarrow$ more trustworthiness)

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- *Archive of Formal Proofs*

Isabelle

```
theorem sqrt2_not_rational:
  "sqrt 2 ∉ℚ "
proof
  let ?x = "sqrt 2"
  assume "?x ∈ℚ "
  then obtain m n :: nat where
    sqrt_rat: "¦?x¦ = m / n" and lowest_terms: "coprime m n"
    by (rule Rats_abs_nat_div_natE)
  hence "m^2 = ?x^2 * n^2" by (auto simp add: power2_eq_square)
  hence eq: "m^2 = 2 * n^2" using of_nat_eq_iff power2_eq_square by fastforce
  hence "2 dvd m^2" by simp
  hence "2 dvd m" by simp
  have "2 dvd n" proof -
    from ‹2 dvd ›m obtain k where "m = 2 * k" ..
    with eq have "2 * n^2 = 2^2 * k^2" by simp
    hence "2 dvd n^2" by simp
    thus "2 dvd n" by simp
  qed
  with ‹2 dvd ›m have "2 dvd gcd m n" by (rule gcd_greatest)
  with lowest_terms have "2 dvd 1" by simp
  thus False using odd_one by blast
qed
```

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- modern syntax (Unicode support)
- Very extensive library Mathlib
- A lot of recent high-impact development
- Functional style; macros for tactics

Lean (v4)

```
theorem and_swap (p q : Prop) : p ∧ q → q ∧ p := by
  intro h
  apply And.intro
  · exact h.right
  · exact h.left
```

Practical part with Lean 4 (feel free to join in yourself!)

Lean online editor

Files