The Category of Iterative Sets in Cubical Agda

Master Thesis, Stockholm University

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Previous article by Gratzer, Gylterud, Mörtberg, Stenholm [1]

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> [...] even though the naturality requires complex path algebra to state properly, in the specific CwF on V all these paths are given by reflexivity.

Idea: Easier to formalize in cubical setting

2. Iterative Sets

Tarski-style Universe

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- A universe $\mathcal{U}$ of **codes**

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Idea: Viewing **sets** as **trees**:

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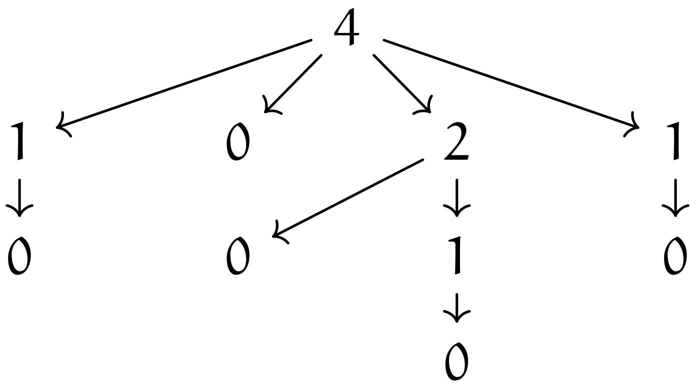
Idea: Viewing **sets** as **trees**:

$$\{\{\emptyset\}, \emptyset, \{\emptyset, \{\emptyset\}\}, \{\emptyset\}\}$$



Iterative Multisets

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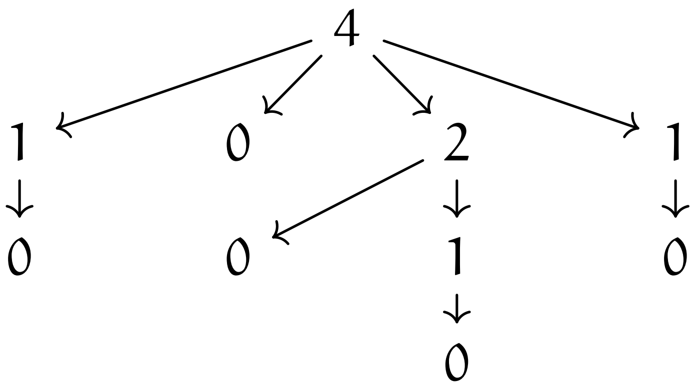

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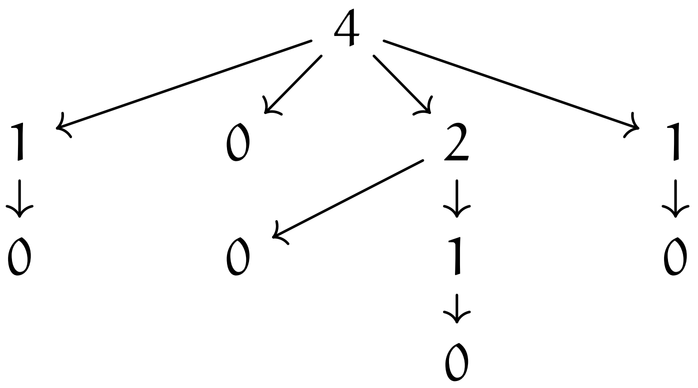

Data: some **type** A and some **function** $f : A \to$ ...

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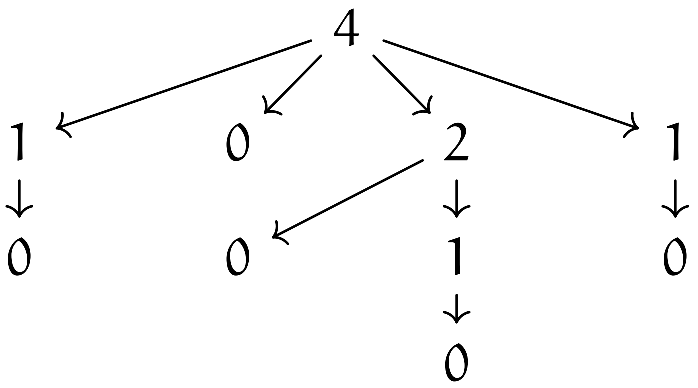

Data: some **type** A and some **function** $f : A \to$ ...

Definition **W-types:** $W_{x:A} B(x)$ with $A : \mathrm{Type}$ and $B : A \to \mathrm{Type}$ has elements

$$\mathrm{sup}(a, f) \quad \textbf{with} \quad a : A, \quad f : B(a) \to W_{x:A} B(x)$$

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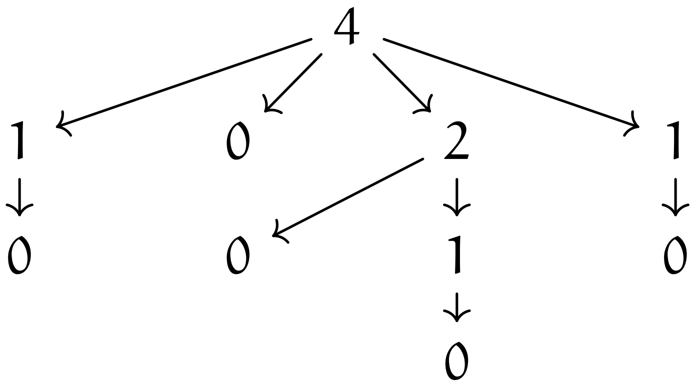

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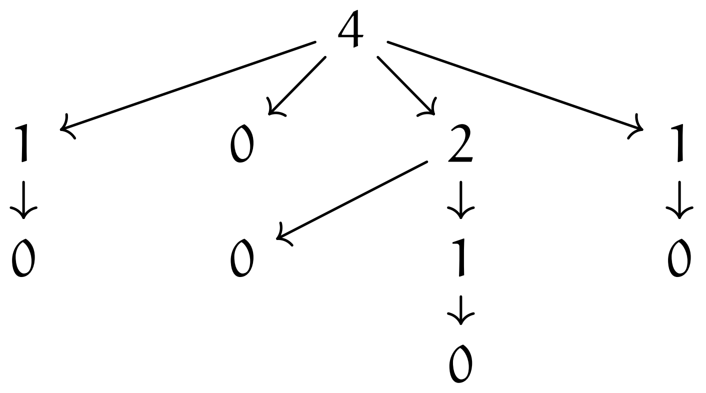

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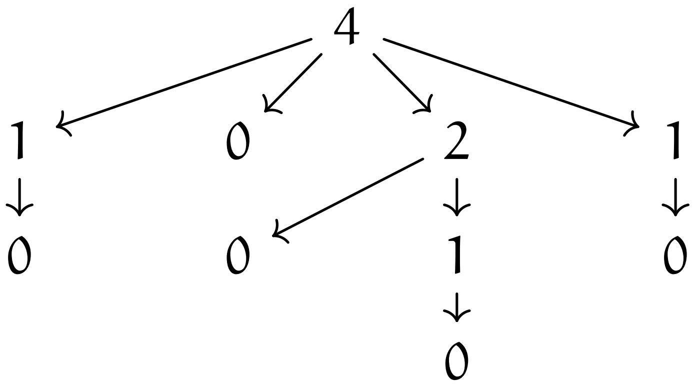

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- **decoding:** $\mathrm{El}(\mathrm{sup}(A, f)) := A$

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  - identity types, Π-types, Σ-types, quotients, universe codes
- V^0 is a **category** closed under many standard **constructions**

3. Iterative Sets as a model of Dependent Type Theory

Categories with Families

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- a **functor** $- \,.\, - : \int \mathrm{Ty} \to \mathcal{C}$
- for each $(\Gamma, A) : \int \mathrm{Ty}$ a **natural equivalence** in $\Delta : \mathcal{C}$:

$$\mathcal{C}(\Delta, \Gamma \,.\, A) \simeq \sum_{\gamma : \mathcal{C}(\Delta, \Gamma)} \mathrm{Tm}(\Delta, A \cdot \gamma)$$

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$$\mathrm{Tm}(\Gamma, S_\Gamma(A, B)) \simeq \sum_{a : \mathrm{Tm}(\Gamma, A)} \mathrm{Tm}(\Gamma, B \cdot \langle 1_\Gamma, a \rangle)$$

4. Formalization

Formalization Steps

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Part 1

- Iterative **multisets**
- Iterative **sets**
- ... as **universe**
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- **CwF** and **Σ-structure** (general)

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  1. Naive
  2. "Cubical"
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Total: ~4700 LOC

Naturality of Σ-structure

$$
\begin{array}{ccccc}
\Gamma' & & \mathrm{Tm}(\Gamma', S_{\Gamma'}(A, B)) & \xrightarrow{\cong} & \sum_{a' : \mathrm{Tm}(\Gamma', A)} \mathrm{Tm}(\Gamma', (B \cdot \langle 1_{\Gamma'}, a' \rangle)) \\
 & & & & \Big\downarrow {\scriptstyle (\mathrm{id}, -[\sigma])} \\
\Big\uparrow {\scriptstyle \sigma} & & \Big\downarrow {\scriptstyle -[\sigma]} & & \sum_{a' : \mathrm{Tm}(\Gamma', A)} \mathrm{Tm}(\Gamma, (B \cdot \langle 1_{\Gamma'}, a' \rangle \cdot \sigma)) \\
 & & & & \Big\| {\scriptstyle \left(\mathrm{id}, \mathrm{subst}_q^{\mathrm{Tm}(\Gamma, -)}(-)\right)} \\
\Gamma & & \mathrm{Tm}(\Gamma, (S_{\Gamma'}(A, B) \cdot \sigma)) & & \sum_{a' : \mathrm{Tm}(\Gamma', A)} \mathrm{Tm}(\Gamma, (B \cdot \langle \sigma, A \rangle \cdot \langle 1_{\Gamma}, a'[\sigma] \rangle)) \\
\Big\| & & {\scriptstyle \mathrm{subst}_p^{\mathrm{Tm}(\Gamma, -)}(-)} \Big\| & & \Big\downarrow {\scriptstyle (-[\sigma], \mathrm{id})} \\
\Gamma & & \mathrm{Tm}(\Gamma, S_{\Gamma}(A \cdot \sigma, B \cdot \langle \sigma, A \rangle)) & \xrightarrow{\cong} & \sum_{a : \mathrm{Tm}(\Gamma, (A \cdot \sigma))} \mathrm{Tm}(\Gamma, (B \cdot \langle \sigma, A \rangle \cdot \langle 1_{\Gamma}, a \rangle))
\end{array}
$$

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Outlook

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- finish formalization

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- finish formalization
- different model structure instead of CwF

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Bibliography

[1] Daniel Gratzer, Håkon Robbestad Gylterud, Anders Mörtberg, and Elisabeth Stenholm, "The category of iterative sets in homotopy type theory and univalent foundations," *Mathematical Structures in Computer Science*, 2024, doi: 10.1017/S0960129524000288.

5. Appendix

$$
\begin{array}{ccccc}
\Delta & & \mathcal{C}(\Delta, \Gamma \,.\, A) & \xrightarrow{\;\cong\;} & \sum_{\tau:\mathcal{C}(\Delta,\Gamma)} \mathrm{Tm}(\Delta, (A \cdot \tau)) \\
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\Big\uparrow \sigma & & \Big\downarrow \sigma^* & & \Big\| {\scriptstyle (\mathrm{id}, \mathrm{transport}_p(-))} \\
 & & & & \sum_{\tau:\mathcal{C}(\Delta,\Gamma)} \mathrm{Tm}(\Delta', (A \cdot (\tau \circ \sigma))) \\
 & & & & \Big\downarrow {\scriptstyle (\sigma^*, \mathrm{id})} \\
\Delta' & & \mathcal{C}(\Delta', \Gamma \,.\, A) & \xrightarrow{\;\cong\;} & \sum_{\tau:\mathcal{C}(\Delta',\Gamma)} \mathrm{Tm}(\Delta', (A \cdot \tau))
\end{array}
$$

$\mathcal{V}^0$ as a CwF $\mathcal{V}$

V^0 as a CwF $\mathcal{V}$

- **objects:** V^0
- **arrows:** $\mathcal{V}(\Gamma, \Delta) \coloneqq \mathrm{El}(\Gamma) \to \mathrm{El}(\Delta)$
- **types:** $\mathrm{Ty}(\Gamma) \coloneqq \mathrm{El}(\Gamma) \to V^0$
- **terms:** $\mathrm{Tm}(\Gamma, A) \coloneqq \prod_{y:\,\mathrm{El}(\Gamma)} \mathrm{El}(A(y))$

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- **terms:** $\mathrm{Tm}(\Gamma, A) \coloneqq \prod_{y : \mathrm{El}(\Gamma)} \mathrm{El}(A(y))$
- **context extension:** $\Gamma \,.\, A \coloneqq \Sigma^0(\Gamma, A)$
- **Σ-structure:** $S_\Gamma(A, B) \coloneqq \lambda x : \mathrm{El}(\Gamma). \quad \Sigma^0(A(x), \lambda a.B(x, a))$